Financial mathematics, MATH /ECO 3900

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1 Practice problems

1. If $dS = \mu S dt + \sigma S dW$, where W is Wiener process. Find the stochastic differential equation satisfied by
   - $f(S) = As$
   - $f(S) = S^n$
2. If there are two assets satisfying the following stochastic differential equations:
$$dS_i = \mu_i S_i + \sigma_i S_i dW_i \text{ for } i = 1, 2.$$
The wiener process dW_i satisfy $\mathcal{E}(dW_i^2) = dt$. The asset price changes are co related with each other
$$\mathcal{E}(dW_i dW_j) = \rho_{ij} dt,$$
where $-1 \leq \rho_{ij} \leq 1$. Derive the Ito's lemma for a function $f(t, S_1, S_2)$. Generalize the result to n assets.
3. Consider
$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}}$$
$$d_1 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{(T - t)}}.$$
Also $N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx$. Show that
$$N^{'}(d_2) = N^{'}(d_1)\frac{S}{E}e^{r(T-t)},$$
where
$$N^{'}(d) = \frac{1}{\sqrt{2\pi}} e^{-d^2/2}.$$
4. Show that
$$SN^{'}(d_1) - Ee^{-r(T-t)}N^{'}(d_2) = 0.$$

5. Show that
$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$
for European call option and
$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$
satisfy
$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

6. Show that $C(S,t)$ and $P(S,t)$ also satisfy the put-call parity.

   Calculate the following Greeks for both European call and Put options

7. $\Delta_{call} = \frac{\partial C}{\partial S}$

8. $\Delta_{put} = \frac{\partial P}{\partial S}$

9. $\Gamma_{call} = \frac{\partial^2 C}{\partial S^2}$

10. $\Gamma_{put} = \frac{\partial^2 P}{\partial S^2}$

11. Assume $r = 0$, calculate $\Theta = -\frac{\partial V}{\partial t}$

12. Calculate $\Theta_{call} = -\frac{\partial C}{\partial t}$

13. Show that $\Theta_{put} = \Theta_{call} - rEe^{-r(T-t)}$

14. Calculate $Vega_{call} = \frac{\partial C}{\partial \sigma}$

15. Calculate $Vega_{put} = \frac{\partial P}{\partial \sigma}$

16. Calculate $\rho_{call} = \frac{\partial C}{\partial r}$

17. Calculate $\rho_{put} = \frac{\partial P}{\partial r}$

18. What is the random walk followed by European call option.

19. Suppose that European calls of all exercise prices are available. regarding S as fixed and E as variable, show that their price $C(E,t)$ satisfy the partial differential equations
$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 C}{\partial E^2} - rE\frac{\partial C}{\partial E} = 0.$$

20. Transform the Black-Scholes partial differential equation with continuous dividend yield D_0
$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0,$$
into the diffusion equation
$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$
Use the transformation $\tau = T - t$, $x = \log(S/E)$, and $v(x,t) = \frac{V(S,t)}{E}$.

21. Assume the Black-Scholes framework. Consider a 6-month European call option, with:
    - stock price is \$100.
    - strike price of the option is \$98.
    - continuously compounded risk-free interest rate is $r = 0.055$.
    - $D_0 = 0.01$.
    - volatility $\sigma = 0.50$.

    Calculate the price of this put option.
    Note: In the case of continuous dividend rate D_0

    $$d_1 = \frac{\log(S/E) + (r - D_0) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

22. Expand $U(x, \tau + \delta\tau)$ and $U(x, \tau - \delta\tau)$ as Taylor series about (x, τ). Deduce the central difference approximation.

23. Develop implicit finite difference Scheme for Black-Scholes partial differential equation.

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25. Derive the partial differential equation for the Asian option, where

    $$\text{payoff} = \max(S(T) - \frac{1}{T}I(t), 0)$$

    and

    $$I(t) = \int_0^T f(S,t)dt.$$