Solution to the practice questions.

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(a) Since $f(s) = As$, we can use Itô's lemma so

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}(ds)^2$$

$$\Rightarrow df = 0 + A(\mu s\,dt + a s\,dW) + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}a^2 s^2\,dt$$

using $(dW)^2 \approx dt$ as $dt \to 0$

$$\Rightarrow df = \mu A s\,dt + a A s\,dW + \frac{1}{2}(0)$$

$$\Rightarrow df = \mu f(s)\,dt + a f(s)\,dW$$

(b)

$$f(s) = s^n$$

By Itô's lemma

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial s}ds + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}(ds)^2$$

$$\Rightarrow df = 0 + n s^{n-1}(\mu s\,dt + a s\,dW) + \frac{1}{2}n(n-1)s^{n-2}\cdot a^2 s^2\,dt$$

$$\Rightarrow df = n\mu s^n dt + n a s^n dW + \frac{1}{2}n(n-1)a^2 s^n dt$$

$$\Rightarrow df = n\mu f(s)\,dt + n a f(s)\,dW + \frac{1}{2}n(n-1)a^2 f(s)\,dt$$

$$\Rightarrow df = \left(n\mu + \frac{1}{2}n(n-1)\right)f(s)\,dt + n a f(s)\,dW$$

② Since $f(S_1, S_2, t)$ then by Tylor theorem

$$f(S_1+dS_1, S_2+dS_2, t+dt) = f(S_1, S_2, t) + \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 + \frac{\partial f}{\partial t} dt$$
$$+ \frac{1}{2} \frac{\partial^2 f}{\partial S_1^2} (dS_1)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial S_2^2} (dS_2)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2$$
$$+ \frac{1}{2} \frac{\partial^2 f}{\partial S_1 \partial S_2} dS_1 dS_2 + \frac{1}{2} \frac{\partial^2 f}{\partial S_1 \partial t} dS_1 dt$$
$$+ \frac{1}{2} \frac{\partial^2 f}{\partial S_2 \partial t} dS_2 dt + \cdots$$

Shifting $f(S_1, S_2, t)$ to LHS and using

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1$$
$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2$$
$$(dW_i)^2 = dt \quad \text{as} \quad dt \to 0$$
$$dW_1 dW_2 = \rho_{ij} dt$$

we arrive at

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_1} (\mu_1 S_1 dt + \sigma_1 S_1 dW_1) + \frac{\partial f}{\partial S_2} (\mu_2 S_2 dt + \sigma_2 S_2 dt)$$
$$+ \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} dt + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} dt$$
$$+ \frac{1}{2} \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} dW_1 dW_2 + \cdots$$

coefficient of $dS_1 dS_2$

$$dS_1 dS_2 = (\mu_1 S_1 dt + \sigma_1 S_1 dW_1)(\mu_2 S_2 dt + \sigma_2 S_2 dW_2)$$
$$= \mu_1 \mu_2 S_1 S_2 dt^2 + \mu_1 \sigma_2 S_1 S_2 dt\, dW_2 + \sigma_1 \mu_2 S_1 S_2 dW_1 dt$$
$$+ \sigma_1 \sigma_2 S_1 S_2 dW_1 dW_2$$

Since $dt \to 0$, so

$$dS_1 dS_2 \simeq \sigma_1 \sigma_2 S_1 S_2 dW_1 dW_2$$

From here

$$df = \frac{\partial f}{\partial t}dt + \left(\mu_1 S_1 \frac{\partial f}{\partial S_1} + \mu_2 S_2 \frac{\partial f}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2}\right)dt$$

$$+\frac{1}{2}\sigma_1\sigma_2 S_1 S_2 \underline{\rho_{12}\, dt}^{(*)} + \sigma_1 S_1 \frac{\partial f}{\partial S_1}dW_1 + \sigma_2 S_2 \frac{\partial f}{\partial S_2}dW_2$$

(*) using $E(dW_1 dW_2) = \rho_{12}\, dt$

$$\Rightarrow df = \frac{\partial f}{\partial t}dt$$

$$+\left(\mu_1 S_1 \frac{\partial f}{\partial S_1} + \mu_2 S_2 \frac{\partial f}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \frac{1}{2}\sigma_1\sigma_2 S_1 S_2 \rho_{12}\right)dt$$

$$+ \sigma_1 S_1 \frac{\partial f}{\partial S_1}dW_1 + \sigma_2 S_2 \frac{\partial f}{\partial S_2}dW_2$$

Or

$$df = \frac{\partial f}{\partial t}dt.$$

$$+\left(\sum_{i=1}^{2}\mu_i S_i \frac{\partial f}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{2}\sigma_i^2 S_i^2 \frac{\partial^2 f}{\partial S_i^2} + \frac{1}{2}\sum_{i,j=1,\, i\neq j}^{2}\sigma_i\sigma_j S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j}\right)dt$$

$$+ \sum_{i=1}^{2}\sigma_i S_i \frac{\partial f}{\partial S_i}dW_i$$

we can generalize this result for $f(S_1, S_2, \dots, S_n, t)$, wher $S_1, S_2, \dots, S_n$ are the prices of n-stocks

$$df = \frac{\partial f}{\partial t}dt + \left(\sum_{i=1}^{n}\mu_i S_i \frac{\partial f}{\partial S_i} + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 S_i^2 \frac{\partial^2 f}{\partial S_i^2} + \frac{1}{2}\sum_{i,j=1,\, i\neq j}^{n}\sigma_i\sigma_j S_i S_j \rho_{ij}\right)dt$$

$$+ \sum_{i=1}^{n}\sigma_i S_i \frac{\partial f}{\partial S_i}dW_i$$

$$\frac{\partial^2 f}{\partial S_i \partial S_j}$$

3 we must show that

$$N'(d_2) = N'(d_1)\,\frac{S}{E}\,e^{r(T-t)}$$

Since $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}\,dx$

$$\Rightarrow N'(d) = \frac{\partial N(d)}{\partial d} = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{d^2}{2}}$$

Consider

$$N'(d_2) = \frac{\partial N(d_2)}{\partial d_2} = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_2^2}{2}}$$

Since we know that $d_2 = d_1 - a\sqrt{T-t}$

so $$N'(d_2) = \frac{1}{\sqrt{2\pi}}\,e^{-(d_1 - a\sqrt{T-t})^2/2}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}\,e^{-\{d_1^2 + a^2(T-t) - 2d_1 a\sqrt{T-t}\}/2}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_1^2}{2} - \frac{1}{2}a^2(T-t) + d_1 a\sqrt{T-t}}$$

$$N'(d_2) = N'(d_1)\cdot e^{ad_1\sqrt{T-t}}\cdot e^{-\frac{1}{2}a^2(T-t)}. \quad \text{—} (A)$$

Since $$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}a^2)(T-t)}{a\sqrt{T-t}}$$

$$\Rightarrow ad_1\sqrt{T-t} = \log(S/E) + (r + \tfrac{1}{2}a^2)(T-t)$$

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using the above result in eq Ⓐ, we arrive at

$$N'(d_2) = N'(d_1)\, e^{\log(S/E) + (r + \cancel{\frac{1}{2}\sigma^2})(T-t)} \cdot \cancel{e^{-\frac{1}{2}\sigma^2(T-t)}}$$

$$N'(d_2) = N'(d_1)\, \frac{S}{E} \cdot e^{r(T-t)} \qquad \because e^{\log(S/E)} = \frac{S}{E}$$

(4) We must show that

$$S N'(d_1) - E e^{-r(T-t)} N'(d_2) = 0$$

From Q.3, we know that

$$N'(d_2) = \frac{S}{E} N'(d_1) \cdot e^{r(T-t)}.$$

Hence

$$S N'(d_1) - \cancel{E}\, \cancel{e^{-r(T-t)}} \cdot \frac{S}{\cancel{E}} N'(d_1)\, \cancel{e^{r(T-t)}}$$

$$\Rightarrow S N'(d_1) - S N'(d_1)$$

$$= 0$$

(5) To show that $c(S,t) = SN(d_1) - E\,e^{-r(T-t)} N(d_2)$ satisfy the Black-scholes partial differential equation, we must show that

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0 \longrightarrow (1)$$

(i) First calculat $\frac{\partial c}{\partial t}$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\left(SN(d_1) - E\,e^{-r(T-t)}N(d_2)\right)$$

$$= S\frac{\partial N(d_1)}{\partial t} - E\frac{\partial}{\partial t}\underbrace{e^{-r(T-t)}N(d_2)}_{\text{Product rule.}}$$

$$= S\frac{\partial N(d_1)}{\partial d_1}\cdot\frac{\partial d_1}{\partial t} - E\left\{r\,e^{-r(T-t)}N(d_2) + e^{-r(T-t)}\frac{\partial N(d_2)}{\partial t}\right\}$$

$$\frac{\partial c}{\partial t} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t} + Ere^{-r(T-t)}N(d_2) + e^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial t}$$

$$\frac{\partial c}{\partial t} = S\,N'(d_1)\frac{\partial d_1}{\partial t} + Ere^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t} \longrightarrow (A)$$

we know that

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\Rightarrow \frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}$$

(ii) Next we calculate $\frac{\partial c}{\partial S}$

$$\frac{\partial c}{\partial S} = \frac{\partial}{\partial S}\left(SN(d_1) - E\,e^{-r(T-t)}N(d_2)\right)$$

$$\Rightarrow \frac{\partial c}{\partial S} = N(d_1) + S\frac{\partial N(d_1)}{\partial S} - E\,e^{-r(T-t)}\frac{\partial N(d_2)}{\partial S}$$

$$\frac{\partial c}{\partial S} = N(d_1) + S\,\frac{\partial N(d_1)}{\partial d_1}\cdot\frac{\partial d_1}{\partial S} - E e^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial S}$$

$$\frac{\partial c}{\partial S} = N(d_1) + S\,N'(d_1)\frac{\partial d_1}{\partial S} - E e^{-r(T-t)} N'(d_2)\frac{\partial d_2}{\partial S}$$

$$\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}\left\{\frac{\log(S/E) \pm (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right\}$$

$$= \frac{1}{S\sigma\sqrt{T-t}}$$

also $\dfrac{\partial^2 d_2}{\partial S^2} = \dfrac{\partial^2 d_1}{\partial S^2}$

(iii) Next we calculate $\dfrac{\partial^2 c}{\partial S^2}$

$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial c}{\partial S}\right) = \frac{\partial}{\partial S}\left\{N(d_1) + S N'(d_1)\frac{\partial d_1}{\partial S} - E e^{-r(T-t)} N'(d_2)\frac{\partial d_2}{\partial S}\right\}$$

$$\Rightarrow \frac{\partial^2 c}{\partial S^2} = \frac{\partial N(d_1)}{\partial S} + \frac{\partial}{\partial S}\left\{S N'(d_1)\frac{\partial d_1}{\partial S}\right\} - \frac{\partial}{\partial S}\left\{E e^{-r(T-t)} N'(d_2)\frac{\partial d_2}{\partial S}\right\}$$

$$\Rightarrow \frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S} + N'(d_1)\left(\frac{\partial d_1}{\partial S}\right) + S N''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2 + S N'(d_1)\frac{\partial^2 d_1}{\partial S^2}$$
$$- E e^{-r(T-t)}\left\{N''(d_2)\left(\frac{\partial d_2}{\partial S}\right)^2 + N'(d_2)\frac{\partial^2 d_2}{\partial S^2}\right\} \longrightarrow (C)$$

Putting eq (A), eq (B) and eq (C) into (1)

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0$$

$$SN'(d_1)\frac{\partial d_1}{\partial S}-E\,\cancel{e^{-r(T-t)}N(d_2)}-E\,e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

$$+\frac{1}{2}\sigma^2 S^2\Big\{N'(d_1)\frac{\partial d_1}{\partial S}+N'(d_1)\frac{\partial d_1}{\partial S}+SN''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2+SN'(d_1)\frac{\partial^2 d_1}{\partial S^2}$$

$$-E\,e^{-r(T-t)}N''(d_2)\left(\frac{\partial d_2}{\partial S}\right)^2-E\,e^{-r(T-t)}N'(d_2)\frac{\partial^2 d_2}{\partial S^2}\Big\}$$

$$+rS\Big\{\cancel{N(d_1)}+SN'(d_1)\frac{\partial d_1}{\partial S}-E\,e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}\Big\}$$

$$-r\Big\{\cancel{SN(d_1)}-E\,\cancel{e^{-r(T-t)}N(d_2)}\Big\}=0 \longrightarrow (A1)$$

From Q.3, we have

$$N'(d_2)=N'(d_1)\frac{S}{E}e^{r(T-t)}$$

Differentiate again w.r.t S

$$N''(d_2)\frac{\partial d_2}{\partial S}=\frac{e^{r(T-t)}}{E}\left(N''(d_1)\frac{\partial d_1}{\partial S}+N'(d_1)\right)$$

$$\Rightarrow N''(d_2)=\frac{e^{r(T-t)}}{E}\left(N''(d_1)\frac{\cancel{\partial d_1/\partial S}}{\cancel{\partial d_2/\partial S}}+\frac{N'(d_1)}{\frac{\partial d_2}{\partial S}}\right)$$

Since $\frac{\partial d_1}{\partial S}=\frac{\partial d_2}{\partial S}$

So

$$N''(d_2)=\frac{e^{r(T-t)}}{E}\left(N''(d_1)+N'(d_1)\frac{1}{\frac{\partial d_1}{\partial S}}\right)$$

From eq (A1)

replacing $\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{a}{2\sqrt{T-t}}$

$$sN'(d_1)\frac{\partial d_1}{\partial t} - E e^{-r(T-t)} N'(d_2)\left(\frac{\partial d_1}{\partial t} + \frac{a}{2\sqrt{T-t}}\right)$$

$$+ a^2 s^2 N'(d_1)\frac{\partial d_1}{\partial s} + \frac{1}{2}a^2 s^3 N''(d_1)\left(\frac{\partial d_1}{\partial s}\right)^2 + \frac{1}{2}a^2 s^3 N'(d_1)\frac{\partial^2 d_1}{\partial s^2}$$

$$-\frac{1}{2}a^2 s^2\left(N''(d_1)\, s + N'(d_1)\frac{1}{\frac{\partial d_1}{\partial s}}\right)\left(\frac{\partial d_1}{\partial s}\right)^2$$

$$-\frac{1}{2}a^2 s^2 N'(d_1)\frac{s}{E}e^{r(T-t)}\frac{\partial^2 d_2}{\partial s^2}$$

$$+ r s^2 N'(d_1)\frac{\partial d_1}{\partial s} - rs\left(N'(d_1)\frac{s}{E}e^{r(T-t)}\right)E e^{-r(T-t)}\frac{\partial d_2}{\partial s}$$

replace $N'(d_2) = N'(d_1)\frac{s}{E}e^{r(T-t)}$, we get

$$sN'(d_1)\cancel{\frac{\partial d_1}{\partial t}} - sN'(d_1)\cancel{\frac{\partial d_1}{\partial t}} - sN'(d_1)\frac{a}{2\sqrt{T-t}}$$

$$\underbrace{+ a^2 s^2 N'(d_1)\frac{\partial d_1}{\partial s}}_{*} \cancel{+\frac{1}{2}a^2 s^3 N''(d_1)\left(\frac{\partial d_1}{\partial s}\right)^2} \cancel{+\frac{1}{2}a^2 s^3 N'(d_1)\frac{\partial^2 d_1}{\partial s^2}}$$

$$\cancel{-\frac{1}{2}a^2 s^3 N''(d_1)\left(\frac{\partial d_1}{\partial s}\right)^2} \underbrace{-\frac{1}{2}a^2 s^2 N'(d_1)\left(\frac{\partial d_1}{\partial s}\right)}_{**} \cancel{-\frac{1}{2}a^2 s^3 N'(d_1)\frac{\partial^2 d_1}{\partial s^2}}$$

$$\cancel{+ r s^2 N'(d_1)\frac{\partial d_1}{\partial s}} \cancel{- r s^2 N'(d_1)\frac{\partial d_1}{\partial s}}$$

$* + **$

$$= -sN'(d_1)\frac{a}{2\sqrt{T-t}} + \frac{1}{2}a^2 s^2 N'(d_1)\frac{\partial d_1}{\partial s}$$

$$= \frac{N'(d_1)}{2}\left\{\frac{-sa}{\sqrt{T-t}} + a^2 s^2\frac{\partial d_1}{\partial s}\right\}$$

$\frac{\partial d_1}{\partial s} = \frac{1}{as\sqrt{T-t}}$

$$= \frac{N'(d_1)}{2}\left\{\frac{-sa}{\sqrt{T-t}} + a^2 s^2 \cdot \frac{1}{as\sqrt{T-t}}\right\}$$

$$= \frac{N'(d_1)}{2}\left\{-\frac{sa}{\sqrt{T-t}} + \frac{as}{\sqrt{T-t}}\right\} = 0$$

This shows that

$$C(s,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

satisfy the Black-scholes partial differential equation.

Similarly, we can also show that

$$P(s,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$

also satisfy Black-scholes partial differential equation.

⑥ We know for put-call parity that

$$C - P = S - Ee^{-r(T-t)}$$

where $C \equiv$ is European call option
$P \equiv$ is European Put option

Consider $C-P$ and plug in the values of C and P

$$\left(SN(d_1) - Ee^{-r(T-t)}N(d_2)\right) - \left(Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\right)$$

$$\Rightarrow S\left(N(d_1) + N(-d_1)\right) - Ee^{-r(T-t)}\left(N(d_2) + N(-d_2)\right) \rightarrow (A)$$

Now consider

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}\,dx$$

$$N(d) + N(-d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d} e^{-x^2/2}\,dx$$

$$N(d)+N(-d)=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{d} e^{-x^2/2}\,dx+\int_{-\infty}^{-d} e^{-x^2/2}\,dx\right)$$

$$\Rightarrow N(d)+N(-d)=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{d} e^{-x^2/2}\,dx-\int_{\infty}^{d} e^{-x^2/2}\,(-dx)\right)$$

Here we changed the signs of upper and lower limits in second integral.

$$\Rightarrow N(d)+N(-d)=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{d} e^{-x^2/2}\,dx+\int_{d}^{\infty} e^{-x^2/2}\,dx\right)$$

$$\Rightarrow N(d)+N(-d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

$$\Rightarrow N(d)+N(-d)=1$$

Hence from eq. (A) on the previous page

$$C-P=S(1)-E\,e^{-r(T-t)}(1)$$

$$\Rightarrow C-P=S-E\,e^{-r(T-t)}$$

Showing that C & P satisfy put-call parity.

⑦ Delta for call option

$$\Delta_{call} = \frac{\partial C}{\partial S}$$

where $\cdot\ C(S,t) = S N(d_1) - E\, e^{-r(T-t)} N(d_2)$

$$\frac{\partial C}{\partial S} = \frac{\partial}{\partial S}(S)\, N(d_1) + S \frac{\partial N(d_1)}{\partial S} - E\, e^{-r(T-t)} \frac{\partial N(d_2)}{\partial S}$$

$$\Rightarrow \frac{\partial C}{\partial S} = N(d_1) + S \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial S} - E\, e^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \cdot \frac{\partial d_2}{\partial S}$$

$$\Rightarrow \frac{\partial C}{\partial S} = N(d_1) + S N'(d_1) \frac{\partial d_1}{\partial S} - E\, e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S}$$

(*)

$$\Rightarrow \frac{\partial C}{\partial S} = N(d_1) + \left(S N'(d_1) - E\, e^{-r(T-t)} N'(d_2)\right) \frac{\partial d_1}{\partial S}$$

(*) since $d_1 = \dfrac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}} = \frac{\partial d_2}{\partial S}$$

We also know that

$$N'(d_2) = N'(d_1)\, \frac{S}{E}\, e^{r(T-t)} \qquad (Q.3)$$

So

$$\frac{\partial C}{\partial S} = N(d_1) + \left(S N'(d_1) - \cancel{E}\, \cancel{e^{-r(T-t)}} \cdot N'(d_1) \frac{S}{\cancel{E}} \cancel{e^{r(T-t)}}\right) \frac{\partial d_1}{\partial S}$$

$$\frac{\partial C}{\partial S} = N(d_1) + \left(S N'(d_1) - S N'(d_1)\right) \frac{\partial d_1}{\partial S}$$

$$\Rightarrow \frac{\partial C}{\partial S} \equiv \Delta_{call} = N(d_1)$$

⑧ We know from Put-call parity

$$C - P = S - E e^{-r(T-t)}$$

Differentiating w.r.t "S".

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

$$\Rightarrow \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1$$

$$\Rightarrow \frac{\partial P}{\partial S} = N(d_1) - 1$$

$$\Rightarrow \Delta_{Put} = \underline{N(d_1) - 1}$$

⑨

$\Gamma_{call} \equiv$ gamma for call option.

$$\Gamma_{call} = \frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right)$$

$$\Rightarrow \Gamma_{call} = \frac{\partial}{\partial S}\left(N(d_1)\right)$$

$$\Rightarrow \Gamma_{call} = \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial S}$$

$$\Rightarrow \Gamma_{call} = N'(d_1) \frac{\partial d_1}{\partial S}$$

Since $\quad N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-x^2/2} \, dx$

$$\Rightarrow \frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$

Also $\quad d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

$$\Rightarrow \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

So $\quad \Gamma_{call} = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \cdot \frac{1}{S\sigma\sqrt{T-t}}$

(10) $\Gamma_{put} \equiv$ gamma for put option

$$\Gamma_{put} = \frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S}\left(N(d_1) - 1\right)$$

$$= \frac{\partial}{\partial S} N(d_1)$$

$$= \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right)$$

$$= \frac{\partial^2 C}{\partial S^2} = \Gamma_{call}$$

$$= \frac{e^{-d_1^2/2}}{\sqrt{2\pi}\, S \cdot \sigma\sqrt{T-t}}$$

$$\Gamma_{put} = \Gamma_{call}$$

(11) consider the black-scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} a^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

$r = 0 \Rightarrow$

$$\frac{\partial V}{\partial t} + \frac{1}{2} a^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

By definition $\Theta = -\frac{\partial V}{\partial t}$, $\Gamma = \frac{\partial^2 V}{\partial S^2}$, so

$$\Theta = \frac{1}{2} a^2 S^2 \Gamma$$

$$\Rightarrow \Theta = \frac{1}{2} a^{\cancel{2}} S^{\cancel{2}} \cdot \frac{e^{-d_1^2/2}}{\sqrt{2\pi(T-t)}\, \cancel{S} \cdot \cancel{a}}$$

$$\Rightarrow \Theta = \frac{aS\, e^{-d_1^2/2}}{2\sqrt{2\pi(T-t)}}$$

(12) Calculate $\Theta_{call} = -\frac{\partial V}{\partial t}$

from Black scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$\Rightarrow -\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$\Rightarrow \Theta_{call} = \frac{1}{2}\sigma^2 S^2 \Gamma_{call} + rS\Delta_{call} - rC$$

$$\Rightarrow \Theta_{call} = \frac{1}{2}\sigma^2 S^2 \frac{N'(d_1)}{S\sigma\sqrt{T-t}} + \cancel{rSN(d_1)} - r(\cancel{SN(d_1)} - Ee^{-r(T-t)}N(d_2))$$

$$\Gamma_{call} = \frac{e^{-d_1^2/2}}{\sqrt{2\pi}\,\sigma S\sqrt{T-t}} = \frac{N'(d_1)}{\sigma S\sqrt{T-t}} = \Gamma_{put}$$

$$\Delta_{call} = N(d_1)$$

$$C = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

So $$\Theta_{call} = \frac{1}{2}\frac{\sigma S N'(d_1)}{\sqrt{T-t}} + rEe^{-r(T-t)}N(d_2)$$

x

(13) we must show that

$$\Theta_{put} = \Theta_{call} - rE e^{-r(T-t)}$$

From Black-scholes equation

$$-\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$\Rightarrow -\frac{\partial P}{\partial t} = \frac{1}{2}\sigma^2 S^2 \Gamma_{put} + rS\,\Delta_{put} - rP$$

$$\Rightarrow \Theta_{put} = \frac{1}{2}\sigma^2 S^2 \frac{N'(d_1)}{\sigma S\sqrt{T-t}} + rS\left(N(d_1)-1\right)$$
$$-r\left(E e^{-r(T-t)} N(-d_2) - S N(-d_1)\right)$$

$$\Rightarrow \Theta_{put} = \frac{N'(d_1)\,\sigma S}{2\sqrt{T-t}} + rS N(d_1) - rS$$
$$- rE e^{-r(T-t)} N(-d_2) + rS N(-d_1)$$

$$\Rightarrow \Theta_{put} = \frac{N'(d_1)\ \sigma S}{2\sqrt{T-t}} + rS\cancel{N(d_1)} - \cancel{rS}$$
$$-rE e^{-r(T-t)}\left(1-N(d_2)\right) + r\cancel{S}\left(1-\cancel{N(d_1)}\right)$$

by using $1-N(d) = N(-d)$

$$\Rightarrow \Theta_{put} = \frac{\sigma S N'(d_1)}{\sqrt{T-t}} - rEe^{-r(T-t)}\left(1 - N(d_2)\right)$$

$$\Theta_{put} = \underbrace{\frac{\sigma S N'(d_1)}{\sqrt{T-t}} + rEe^{-r(T-t)} N(d_2)} - rEe^{-r(T-t)}$$

$$\Theta_{put} = \Theta_{call} - rEe^{-r(T-t)}$$

---

(14)

$$\text{Vega for call option} = \frac{\partial C}{\partial \sigma}$$

$$\text{Vega}_{call} = \frac{\partial}{\partial \sigma}\left(SN(d_1) - Ee^{-r(T-t)}N(d_2)\right)$$

$$= S\frac{\partial N(d_1)}{\partial \sigma} - Ee^{-r(T-t)}\frac{\partial N(d_2)}{\partial \sigma}$$

$$= S\frac{\partial N(d_1)}{\partial d_1}\cdot\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial \sigma}$$

$$= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$

$$\Rightarrow \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}$$

$$\Rightarrow \text{Vega}_{call} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - \left(Ee^{-r(T-t)}N'(d_2)\right)\left(\frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}\right)$$

$$Vega_{call} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - \left(\cancel{E}\,\cancel{e^{-r(T-t)}} \cdot N'(d_1)\frac{S}{\cancel{E}}\cancel{e^{+r(T-t)}}\right)\left(\frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}\right)$$

$$N'(d_2) = N'(d_1)\frac{S}{E}e^{r(T-t)}$$

$$\Rightarrow Vega_{call} = \cancel{SN'(d_1)\frac{\partial d_1}{\partial s}} - \cancel{SN'(d_1)\frac{\partial d_1}{\partial \sigma}} + SN'(d_1)\sqrt{T-t}$$

$$Vega_{call} = SN'(d_1)\sqrt{T-t}$$

(15) Vega for put options

From put call parity

$$P - C = Ee^{-r(T-t)} - S$$

Differentiating w.r.t "σ", we get

$$\frac{\partial P}{\partial \sigma} - \frac{\partial C}{\partial \sigma} = 0$$

$$\Rightarrow \frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma}$$

ie $Vega_{call} = Vega_{put}$.

(16) Rho for call options

$$rho_{call} = \frac{\partial c}{\partial r}$$

$$= \frac{\partial}{\partial r}\left(SN(d_1) - E e^{-r(T-t)} N(d_2) \right)$$

$$= S \frac{\partial N(d_1)}{\partial r} - E\left(-(T-t) e^{-r(T-t)} N(d_2) + e^{-r(T-t)} \frac{\partial N(d_2)}{\partial r} \right)$$

$$\Rightarrow rho_{call} = S \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial r} + E(T-t) e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \cdot \frac{\partial d_2}{\partial r}$$

$$\Rightarrow rho_{call} = S N'(d_1) \frac{\partial d_1}{\partial r} + E(T-t) e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial r}$$

Now $$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{\partial d_1}{\partial r} = \frac{1}{\sigma\sqrt{T-t}} \times (T-t)$$

$$\Rightarrow \frac{\partial d_1}{\partial r} = \frac{\sqrt{T-t}}{\sigma}$$

Similarly $$\frac{\partial d_2}{\partial r} = \frac{\sqrt{T-t}}{\sigma}$$

So $rho_{call} = \left(SN'(d_1) \cancel{- E e^{-r(T-t)} N'(d_2)}^{\,0}\right) \frac{\partial d_1}{\partial r}$

$+ E(T-t) N(d_2) e^{-r(T-t)}$

$$rho_{call} = E(T-t) N(d_2) e^{-r(T-t)}$$

(17) $rho_{put.} \equiv \rho_{put} = \frac{\partial P}{\partial r}$

From put-call parity

$$P - C = E e^{-r(T-t)} - S$$

$$\frac{\partial P}{\partial r} - \frac{\partial C}{\partial r} = -E(T-t) e^{-r(T-t)}$$

$$\Rightarrow \frac{\partial P}{\partial r} = \frac{\partial C}{\partial r} - E e^{-r(T-t)} (T-t)$$

$$\Rightarrow \frac{\partial P}{\partial r} = E(T-t) e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} (T-t)$$

$$\Rightarrow \frac{\partial P}{\partial r} = E(T-t) e^{-r(T-t)} \left(N(d_2) - 1\right)$$

$$\Rightarrow \frac{\partial P}{\partial r} = E(T-t) e^{-r(T-t)} N(-d_2)$$

(18)

Let $C(s,t)$ is European call option. By Ito's lemma

$$dc = \frac{\partial c}{\partial t} dt + \frac{\partial c}{\partial s} ds + \frac{1}{2} \frac{\partial^2 c}{\partial s^2} ds^2$$

$$\Rightarrow dc = \frac{\partial c}{\partial t} dt + \frac{\partial c}{\partial s} (\mu s\, dt + a s\, dw) + \frac{1}{2} \frac{\partial^2 c}{\partial s^2} (a^2 s^2 dt)$$

$$\Rightarrow dc = \frac{\partial c}{\partial t} dt + \mu s \frac{\partial c}{\partial s} dt + a s \frac{\partial c}{\partial s} dw + \frac{1}{2} a^2 s^2 \frac{\partial^2 c}{\partial s^2} dt$$

$$\Rightarrow dc = \left(\frac{\partial c}{\partial t} + \mu s \frac{\partial c}{\partial s} + \frac{1}{2} a^2 s^2 \frac{\partial^2 c}{\partial s^2} \right) dt + a s \frac{\partial c}{\partial s} dw$$

Since $c(s,t)$ satisfy Black-scholes equation

so

$$\frac{\partial c}{\partial t} + \frac{1}{2} a^2 s^2 \frac{\partial^2 c}{\partial s^2} + rs \frac{\partial c}{\partial s} - rc = 0$$

$$\Rightarrow \frac{\partial c}{\partial t} + \frac{1}{2} a^2 s^2 \frac{\partial^2 c}{\partial s^2} = rc - rs \frac{\partial c}{\partial s} dw$$

So $$dc = \left(rc - rs \frac{\partial c}{\partial s} + \mu s \frac{\partial c}{\partial s} \right) dt + a s \frac{\partial c}{\partial s} dw$$

$$dc = \left(rc - (r - \mu) s \frac{\partial c}{\partial s} \right) dt + a s \frac{\partial c}{\partial s} dw$$

(19) we must show that $C(E,t)$ satisfy

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial E^2} - rC\frac{\partial C}{\partial E} = 0$$

$$C(E,t) = SN(d_1) - E e^{-r(T-t)} N(d_2)$$

$$\frac{\partial C}{\partial E} = S\frac{\partial N(d_1)}{\partial E} - e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} \frac{\partial N(d_2)}{\partial E}$$

$$\Rightarrow \frac{\partial C}{\partial E} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial E} - e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial E}$$

$$\text{So } \frac{\partial C}{\partial E} = S N'(d_1)\frac{\partial d_1}{\partial E} - e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} N'(d_2)\frac{\partial d_2}{\partial E} \rightarrow (1)$$

Since

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{\partial d_1}{\partial E} = \frac{\partial}{\partial E}\left(\frac{\log S - \log E}{\sigma\sqrt{T-t}} + \frac{(r+\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-t}\right)$$

$$\frac{\partial d_1}{\partial E} = \frac{-1}{E\sigma\sqrt{T-t}}$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$

so $\frac{\partial d_2}{\partial E} = \frac{\partial d_1}{\partial E}$

$\Rightarrow$ from eq (1)

$$\frac{\partial C}{\partial E} = \left(SN'(d_1) - Ee^{-r(T-t)}N'(d_2)\right)\frac{\partial d_1}{\partial E} - e^{-r(T-t)}N(d_2) \rightarrow (2)$$

we have shown this in Q3, Q4

we know that

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-x^2/2}\, dx$$

$$N'(d_2) = \frac{\partial N(d_2)}{\partial d_2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}$$

$$\Rightarrow N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-(d_1 - a\sqrt{T-t})^2/2}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \frac{a^2(T-t)}{2} - d_1 a(\sqrt{T-t})}$$

$$N'(d_2) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}\right) e^{\frac{a^2(T-t)}{2}} \cdot e^{-d_1 a\sqrt{T-t}}$$

Since $a\sqrt{T-t}\, d_1 = \log(S/E) + (r + \frac{1}{2}a^2)(T-t)$

So $e^{-d_1 a\sqrt{T-t}} = \frac{S}{E} e^{r(T-t)} \cdot e^{\frac{a^2(T-t)}{2}}$

Now

$$N'(d_2) = N'(d_1) \frac{S}{E} e^{r(T-t)}$$

Now $S N'(d_1) - E e^{-r(T-t)} N'(d_2)$

$$= S N'(d_1) - \cancel{E}\, \cancel{e^{-r(T-t)}} \cdot N'(d_1) \frac{S}{\cancel{E}} \cancel{e^{r(T-t)}}$$

$$= S N'(d_1) - S N'(d_1)$$

$$= 0$$

From eq (2) $\frac{\partial c}{\partial E} = -e^{-r(T-t)} N(d_2)$ —— (A)

Next we calculate $\frac{\partial^2 c}{\partial E^2}$

$$\frac{\partial^2 c}{\partial E^2} = \frac{\partial}{\partial E}\left(-e^{-r(T-t)} N(d_2)\right)$$

$$= -e^{-r(T-t)} \frac{\partial N(d_2)}{\partial E}$$

$$= -e^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \cdot \frac{\partial d_2}{\partial E}$$

$$= -e^{-r(T-t)} N'(d_2) \frac{1}{E\sigma\sqrt{T-t}} \leftarrow \frac{\partial d_1}{\partial E}$$

so $$\frac{\partial^2 c}{\partial E^2} = -\frac{e^{-r(T-t)} N'(d_2)}{E\sigma\sqrt{T-t}}$$ —— (B)

Lastly, we calculate $\frac{\partial c}{\partial t}$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\left[S N(d_1) - E e^{-r(T-t)} N(d_2)\right]$$

$$\frac{\partial c}{\partial t} = S N'(d_1) \frac{\partial d_1}{\partial t} - E\left(r e^{-r(T-t)} N(d_2) + e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial t}\right)$$

$$= S N'(d_1) \frac{\partial d_1}{\partial t} - E r e^{-r(T-t)} N(d_2) + E e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial t}$$

using $\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{T-t}}$ &

$N'(d_2) = N'(d_1) \frac{S}{E} e^{r(T-t)}$

we arrive at

$$\frac{\partial c}{\partial t} = \frac{-\sigma S N'(d_1)}{2\sqrt{T-t}} - E e^{-r(T-t)} N(d_2)$$

$\hookrightarrow$ (C)

From eq (A), (B) and (C)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 C}{\partial E^2} - rE\frac{\partial C}{\partial E}$$

$$= \frac{-\sigma S N'(d_1)}{2\sqrt{T-t}} \cancel{- Ere^{-r(T-t)} N(d_2)} + \frac{1}{2}\frac{\cancel{\sigma^2} \cancel{E^2} e^{-r(T-t)} N'(d_2)}{\cancel{\sigma}\cancel{E}\sqrt{T-t}}$$

$$\cancel{+ rE e^{-r(T-t)} N(d_2)}$$

$$= \frac{-\sigma S N'(d_1)}{2\sqrt{T-t}} + \frac{1}{2}\frac{\sigma E e^{-r(T-t)}}{\sqrt{T-t}} N'(d_2)$$

$$= \frac{-\sigma S N'(d_1)}{2\sqrt{T-t}} + \frac{\sigma \cancel{E} \cancel{e^{-r(T-t)}}}{2\sqrt{T-t}} \times N'(d_1)\frac{S}{\cancel{E}} \cancel{e^{r(T-t)}}$$

$$= \frac{-\sigma S \cancel{N'(d_1)}}{2\cancel{\sqrt{T-t}}} + \frac{\sigma S \cancel{N'(d_1)}}{2\cancel{\sqrt{T-t}}}$$

$$= 0$$

So the given PDE is satisfied.

(20) The given differential equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0 \longrightarrow (A)$$

since $\tau = T - t$, $x = \ln(S/E)$, $u(x,\tau) = \dfrac{V(x,t)}{E}$

$$\Rightarrow t = T - \tau, \quad S = E e^{x}, \quad V(x,t) = E\,u(x,\tau)$$

Find $\dfrac{\partial V}{\partial t}$

$$\frac{\partial V}{\partial t} = \frac{\partial E u}{\partial t} = E \frac{\partial u}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = -E \frac{\partial u}{\partial \tau}$$

Find $\dfrac{\partial V}{\partial S}$

$$\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}(E u) = E \frac{\partial u}{\partial S} = E \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial S}$$

$$\Rightarrow \frac{\partial V}{\partial S} = E \frac{\partial u}{\partial x} \cdot \frac{1}{S} \Rightarrow \boxed{\frac{\partial V}{\partial S} = \frac{E}{S} \frac{\partial u}{\partial x}}$$

Find $\dfrac{\partial^2 V}{\partial S^2}$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{E}{S} \frac{\partial u}{\partial x}\right)$$

$$= E\left(-\frac{1}{S^2} \frac{\partial u}{\partial x} + \frac{1}{S^2} \frac{\partial^2 u}{\partial x^2}\right)$$

Plugin values of $\dfrac{\partial V}{\partial t}$, $\dfrac{\partial V}{\partial S}$ & $\dfrac{\partial^2 V}{\partial S^2}$ into (A)

$$-\cancel{E} \frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2 \cancel{S^2} \cancel{E}\left(-\frac{1}{\cancel{S^2}} \frac{\partial u}{\partial x} + \frac{1}{\cancel{S^2}} \frac{\partial^2 u}{\partial x^2}\right) + (r - D_0) \cancel{S} \frac{\cancel{E}}{\cancel{S}} \frac{\partial u}{\partial x} - r\cancel{E}u = 0$$

$$\Rightarrow \frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\sigma^2 \frac{\partial u}{\partial x} + (r - D_0) \frac{\partial u}{\partial x} - ru = 0$$

$$\Rightarrow \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} + \left\{ \frac{(r-D_0)}{\frac{1}{2}\sigma^2} \right\} \frac{\partial v}{\partial x} - \left\{ \frac{r}{\frac{1}{2}\sigma^2} \right\} v = 0$$

$$\Rightarrow \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{r-D_0}{\frac{1}{2}\sigma^2} - 1 \right) \frac{\partial v}{\partial x} - \left(\frac{r}{\frac{1}{2}\sigma^2} \right) v = 0$$

let $k' = \dfrac{r-D_0}{\frac{1}{2}\sigma^2}$, $k = \dfrac{r}{\frac{1}{2}\sigma^2}$

So $$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k'-1) \frac{\partial v}{\partial x} - k v = 0 \qquad \text{——— (B)}$$

Next let $v(x,\tau) = e^{\alpha x + \beta \tau} u(x,\tau)$

$$\frac{\partial v}{\partial \tau} = e^{\alpha x + \beta \tau} \left(\frac{\partial u}{\partial \tau} + \beta u \right)$$

$$\frac{\partial v}{\partial x} = e^{\alpha x + \beta \tau} \frac{\partial u}{\partial x} + \alpha e^{\alpha x + \beta \tau} u = e^{\alpha x + \beta \tau} \left(\frac{\partial u}{\partial x} + \alpha u \right)$$

$$\frac{\partial^2 v}{\partial x^2} = e^{\alpha x + \beta \tau} \left(\alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \right)$$

Plugin these values in eq (B), after some cancellation

$$\frac{\partial u}{\partial \tau} + \beta u = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k'-1)\left(\frac{\partial u}{\partial x} + \alpha u \right) - k u$$

$$\Rightarrow \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (2\alpha + k' - 1) \frac{\partial u}{\partial x} + (\alpha^2 + (k'-1)\alpha - \beta - k) u$$

If we select

$$2\alpha + k' - 1 = 0$$

$$\Rightarrow \alpha = \frac{1-k'}{2}$$

and $\beta = \alpha^2 + (k'-1)\alpha - k$

then

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

21 Here $T-t = \frac{6}{12} = 0.5$

$\delta = 0.01$

$\sigma = 0.05$

$r = 0.55$

$S = 100$

$E = 98$

According to Black-scholes formula

$$P(s,t) = E e^{-r(T-t)} N(-d_2) - S N(-d_1)$$

Now

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\log(100/98) + (0.05 + \frac{1}{2}\sigma^2)\,0.5}{\sigma\sqrt{T-t}}$$

$$d_1 = -0.297558191$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-x^2/2}\, dx$$

$$N(-0.297558191) = 0.3830$$

This can be calculated in MATLAB as

P = normcdf(-0.29755819l)

Now $N(-d_2) = N(-d_1 + \sigma\sqrt{T-t})$
$= N(0.0559) = 0.5223$

Now $P(S,t) = E e^{-r(T-t)} N(-d_2) - S N(-d_1)$
$= 98\, e^{-0.055 \times 0.5} \times 0.5223 - 100 \times 0.38302$
$= 11.688$

(22) $u(x, \tau + \delta\tau) = u(x,\tau) + \frac{\partial u}{\partial \tau}\delta\tau + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}\delta\tau^2 + \cdots$

$u(x, \tau - \delta\tau) = u(x,\tau) - \frac{\partial u}{\partial \tau}\delta\tau + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}\delta\tau^2 + \cdots$

Subtracting we get

$u(x, \tau + \delta\tau) - u(x, \tau - \delta\tau) \approx 2 \frac{\partial u}{\partial \tau}\delta\tau$

$$\Rightarrow \frac{\partial u}{\partial \tau} \simeq \frac{u(x, \tau+\delta\tau) - u(x, \tau - \delta\tau)}{2\delta\tau}$$

(23) See the lectures ?

(24) See the lecture ?

(25) Payoff is $\max\left(S(T) - \frac{1}{T}\int_0^T f(s,t)\,dt,\ 0\right)$

let us define $I = \int_0^T f(s,t)\,dt$

$\Rightarrow dI = f(s,t)\,dt$

Let $V(S,I,t)$ be an asian option

applying Ito's lemma to $V(S,I,t)$

$$V(S+dS,\ I+dI,\ t+dt) = V(S,I,t) + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial I}dI + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{(\partial I)^2}(dI)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \cdots$$

re-arranging we get

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial I}dI + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{(\partial I)^2}(dI)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2$$

we know that

$dS = \mu S\,dt + \sigma S\,dW$

$\Rightarrow (dS)^2 = \sigma^2 S^2\,dt$, as $dt \to 0$

so $dV = \frac{\partial V}{\partial S}(\mu S\,dt + \sigma S\,dW) + \frac{\partial V}{\partial I} f(s,t)\,dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\,dt$

Let us build a risk-less portfolio by being long on an option and short on the underlying asset

$$\Pi = V - \Delta S$$

$$\Rightarrow d\Pi = dV - \Delta dS$$

$$\Rightarrow d\Pi = \left(\mu S \frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial I} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW - \Delta \mu S\, dt - \Delta \sigma S\, dW \tag{1}$$

Since Π is risk-less, so

$$\left(\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S\right) dW = 0$$

$$\Rightarrow \boxed{\frac{\partial V}{\partial S} = \Delta}$$

If we plugin $\frac{\partial V}{\partial S} = \Delta$ into eq(1), we get

$$d\Pi = \left(f(S,t)\frac{\partial V}{\partial I} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

Π is risk-less portfolio, so

$$d\Pi = r\Pi\, dt$$

$$\text{so } r\Pi\, \cancel{dt} = \left(f(S,t)\frac{\partial V}{\partial I} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\cancel{dt}$$

$$\Rightarrow r\Pi = f(S,t)\frac{\partial V}{\partial I} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

$$\Rightarrow r\left(V - \frac{\partial V}{\partial S} S\right) = f(S,t)\frac{\partial V}{\partial I} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial I} - rV = 0$$